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# On the Spectra of Infinite-Dimensional Jacobi Matrices

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The Green's function method used by Case and Kac is extended to include unbounded Jacobi matrices. As a first application an upper bound on the number of eigenvalues is calculated, using the method of Bargmann. Another bound is found using the Birman–Schwinger argument, which is valid for matrix orthogonal polynomials. © 1988 Academic Press, Inc.

## I. INTRODUCTION

Let $l_2 \supset D(J) \xrightarrow{J} l_2$ be a self-adjoint operator with the representation

$$Je_n = a(n+1)\,e_{n+1} + b(n)\,e_n + a(n)\,e_{n-1}, \qquad n = 1, 2, ... \tag{I.1}$$

$$Je_0 = a(1)\,e_1 + b(0)\,e_0. \tag{I.2}$$

The spectrum of $J$, $\sigma(J)$, is the set of all $x$ such that $(J - xI)^{-1}$ is not a bounded linear operator on $l_2$, and $\sigma_P(J) \subset \sigma(J)$ is the set of all $x$ such that $(J - xI)^{-1}$ is not defined. $\sigma_{\text{ess}}(J)$ (Reed and Simon [18]), the essential spectrum of $J$, is the set of all real $\lambda$ for which $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(J)$ is infinite-dimensional for all $\varepsilon > 0$. Here $P_r = \chi_\Omega(J)$ is a spectral projection of $J$ onto $\Omega$, $\Omega$ a Borel subset of $R$. Let $p(\lambda, n)$ be the orthonormal polynomials associated with $J$ and for each $n$ let $\lambda_{n1} < \lambda_{n2} < \lambda_{n3} < \cdots < \lambda_{nn}$ be the zeros of $p(\lambda, n)$. Setting $\rho(J) = \lim_{i\to\infty} \lim_{n\to\infty} \lambda_{n,i}$ and $\tau(J) = \lim_{j\to\infty} \lim_{n\to\infty} \lambda_{n,n-j+1}$, one finds that $\sigma_{\text{ess}}(J) \subset [\rho, \tau]$ with $\rho$ and $\tau$ being, respectively, the largest and smallest points in $\sigma_{\text{ess}}(J)$ [4].

A question that has been of recent interest (Geronimo and Case [14], Chihara [7–9], Chihara and Nevai [10], and Geronimo [13]) is can one obtain bounds on the number of eigenvalues of $J$ in $[\sigma, \tau]^c$? In [13] an upper bound on the number of eigenvalues of $J$ is given when $J$ is a bounded operator, using an argument first developed by Bargmann [2]. Here we extend the argument to unbounded operators and use a different argument due to Birman [3] and Schwinger [21] to obtain other bounds.



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We proceed as follows: in Section II we construct a general comparison equation using the Green's function, which allows us (Section III) to obtain an upper bound on the number of eigenvalues of $J$ using Bargmann's argument. In Section IV a modification of the Birman–Schwinger argument due to Fonda and Ghirardi [11] is used which gives an alternative upper bound on the number of eigenvalues of $J$. This bound is valid even if the entries in $J$ are themselves matrices.

## II. Construction of the Comparison Equation

Given $a^0(n+1)$, $b^0(n) \in C$, $a^0(n+1) \neq 0$ for all $n \geqslant 0$ we construct the unique solution to the equation

$$a^0(n+1)\, p^0(\lambda, n+1) + b^o(n)\, p^0(\lambda, n) + a^0(n)\, p^0(\lambda, n-1) \\ = \lambda p^0(\lambda, n), \quad n = 0, 1, 2, ..., \tag{II.1}$$

satisfying the initial conditions

$$p^0(\lambda, 0) = 1, \qquad p^0(\lambda, -1) = 0. \tag{II.2}$$

With these polynomials we now construct the unique (Green's function) solution to the equation

$$a^0(n+1)\, G_1(\lambda, n+1, m) + b^0(n)\, G_1(\lambda, n, m) + a^0(n)\, G_1(\lambda, n-1, m) \\ - \lambda G_1(\lambda, n, m) = \delta_{n,m}, \qquad m \geqslant -1, n \geqslant 0, \tag{II.3}$$

with boundary conditions

$$G_1(\lambda, n, m) = 0, \qquad n \geqslant m. \tag{II.4}$$

The solution is (Atkinson [1])

$$G_1(\lambda, n, m) = \begin{cases} 0, & n \geqslant m \\ \dfrac{p_1^0(\lambda, m)\, p^0(\lambda, n) - p_1^0(\lambda, n)\, p^0(\lambda, m)}{W[p_1^0, p^0]}, & -1 \leqslant n < m, \end{cases} \tag{II.5}$$

where $p_1^0(\lambda, m)$ is another solution of (II.1) which is linearly independent of $p^0(\lambda, m)$ and $W[p_1^0, p^0]$ is the Wronskian of $p_1^0$ and $p^0$, i.e.,

$$W[p_1^0, p^0] = a^0(n+1)[p_1^0(\lambda, n+1)\, p^0(\lambda, n) - p^0(\lambda, n)\, p^0(\lambda, n+1)], \tag{II.6}$$

which is independent of $n$ (Case [5]).

There are two representations of $G_1(\lambda, n, m)$ which we will need later and in order to exhibit these representations we introduce other solutions of (II.1). To this end, let $p^{(k)}(\lambda, n)$, $k \geqslant 0$, be the solution of

$$a^0(n+k+1)\, p^{(k)}(\lambda, n+1) + b^0(n+k)\, p^{(k)}(\lambda, n) + a^0(n+k)\, p^{(k)}(\lambda, n-1) = \lambda p^{(k)}(\lambda, n), \qquad n = 0, 1, 2, ..., \tag{II.7}$$

satisfying the initial conditions

$$p^{(k)}(\lambda, 0) = 1, \qquad p^{(k)}(\lambda, -1) = 0. \tag{II.8}$$

In the special case where the polynomials $\{p^0(\lambda, n)\}$ are orthogonal with respect to a unique measure $u^0$ supported on $R$ we define the functions of the second kind $Q^0(\lambda, n)$ as

$$Q^0(\lambda, n) = \int_s \frac{p^0(\lambda, n)}{\lambda - x}\, du^0(x), \qquad n = 0, 1, 2, ..., \lambda \notin s, \tag{II.9}$$

where $s$ is the support of $u^0$. An important property of $Q^0(\lambda, n)$ is that $\{Q^0(\lambda, n)\} \in l_2$ for $\lambda \notin 0(\lambda)$.

LEMMA (II.1). *$G_1(\lambda, n, m)$ has the representation*

$$a^0(n+1)\, G_1(\lambda, n, m) = \begin{cases} 0, & n \geqslant m \\ p^{(n+1)}(\lambda, m-n-1), & -1 \leqslant n < m. \end{cases} \tag{II.10}$$

*Furthermore if the moment problem is determined $G_1(\lambda, n, m)$ can also be represented by*

$$G_1(\lambda, n, m) = \begin{cases} 0, & n \geqslant m \\ Q^0(\lambda, n)\, p^0(\lambda, m) - Q^0(\lambda, m)\, p^0(\lambda, n), & 0 \leqslant n < m. \end{cases} \tag{II.11}$$

*Proof.* From (II.3), (II.4), and (II.8) one has that

$$a^0(n+1)\, G_1(\lambda, n, n+1) = 1 = p^{(n+1)}(\lambda, 0). \tag{II.12}$$

Setting $m = n + l$ in (II.3) and then substituting (II.10) into (II.3), we find that the lemma will be demonstrated if it is shown that

$$p^{(n)}(\lambda, l) = \left(\frac{\lambda - b(n+1)}{a(n+1)}\right) p^{(n+1)}(\lambda, l-1) - \frac{a(n+1)}{a(n+2)}\, p^{(n+2)}(\lambda, l-2), \qquad l = 1, 2, .... \tag{II.13}$$

But from (II.7) we see that $p^{(n)}(\lambda, l)$, $p^{(n+1)}(\lambda, l-1)$, and $p^{(n+2)}(\lambda, l-2)$ satisfy a three-term recurrence formula having the same coefficients. Therefore they are not linearly independent and we can write $p^{(n)}(\lambda, l) =$

$Ap^{(n+1)}(\lambda, l-1) + Bp^{(n+2)}(\lambda, l-2)$, where $A$ and $B$ are independent of $l$. $A$ and $B$ can now be obtained by setting $l=1$ and $l=2$ in the above equation and solving the resulting linear system. To prove the second part we note that given (II.9) we find from (II.7) that $a^0(0)\ Q^0(\lambda, -1) = 1$, and consequently that $W[Q^0, p^0] = -1$. This implies that $Q^0(\lambda, n)$ is linearly independent of $p^0(\lambda, n)$ and (II.11) follows from (II.5).

Given another system of polynomials $\{p(\lambda, n)\}$, satisfying the equation

$$a(n+1)\ p(\lambda, n+1) + b(n)\ p(\lambda, n) + a(n)\ p(\lambda, n-1) = \lambda p(\lambda, n), \qquad n = 0, 1, 2, \ldots \tag{II.14}$$

with initial conditions

$$p(\lambda, 0) = 1, \qquad p(\lambda, -1) = 0 \tag{II.15}$$

and with $a(n+1)$, $b(n) \in C$, $a(n+1) \neq 0$ for all $n \geqslant 0$, we seek to express the above polynomials in terms of the (0) system. To this end, multiplying (II.14) by $\alpha_n = \prod_{i=1}^{n} (a(i)/a^0(i))$, $\alpha(0) = 1$, and setting

$$\hat{p}(\lambda, n) = \alpha_n\, p(\lambda, n), \tag{II.16}$$

we find

$$a^0(n+1)\ \hat{p}(\lambda, n+1) + b(n)\ \hat{p}(\lambda, n) + \frac{a(n)^2}{a^0(n)}\ \hat{p}(\lambda, n-1) = \lambda \hat{p}(\lambda, n), \qquad n = 0, 1, 2, \ldots, \tag{II.17}$$

where $a^0(0) \equiv 1$. Multiplying (II.7) by $\hat{p}(\lambda, n)$ and (II.17) by $G_1(\lambda, n, m)$, subtracting one from the other, and then summing on $n$ from $n = j$ to $n = \infty$ gives the equation

$$\hat{p}(\lambda, m) = a^0(j)\ G_1(\lambda, j-1, m)\ \hat{p}(\lambda, j) - \frac{a(j)^2}{a^0(j)}\ G_1(\lambda, j, m)\ \hat{p}(\lambda, j-1) + \sum_{n=j}^{m-1} K(n, m, \lambda)\ \hat{p}(\lambda, n), \qquad m = j, j+1, \ldots, \tag{II.18}$$

where

$$K(n, m, \lambda) = (b^0(n) - b(n))\ G_1(\lambda, n, m) + a^0(n+1)\left(1 - \frac{a(n+1)^2}{a^0(n+1)^2}\right) G_1(\lambda, n+1, m). \tag{II.19}$$

Thus we have shown

THEOREM (II.1). *Given an arbitrary set of polynomials satisfying* (II.14) *and* (II.15) *with* $a(n+1),\ b(n) \in C,\ a(n+1) \neq 0,\ n \geqslant 0,$ *the scaled polynomials given by* (II.17) *satisfy* (II.18), *where* $G_1(\lambda, n, m)$ *is the solution of* (II.3) *and* (II.4), *with* $a^0(n+1),\ b^0(n) \in C,\ a^0(n+1) \neq 0,\ n \geqslant 0,$ *and* $a^0(0) \equiv 1$.

## III. AN UPPER BOUND ON THE NUMBER OF EIGENVALUES OF $J$

Let $J: D(J) \to l_2$, where $D(J)$ is the domain of $J$, be a self-adjoint operator with the representation

$$\begin{aligned} Je_n &= a(n+1)\, e_{n+1} + b(n)\, e_n + a(n)\, e_{n-1}, \qquad n = 1, 2, \ldots \\ Je_0 &= a(1)\, e_1 + b(0)\, e_0. \end{aligned} \tag{III.1}$$

Here $\{e_n\}$ is the natural basis in $l_2$, and $a(i) > 0$, $i > 0$. The problem we are interested in is can we find an upper bound on the number of eigenvalues of $J$ (the number of solutions of $J\psi = \lambda\psi$, $\psi \in D(J)$) that lie above the essential spectrum $\sigma_{\text{ess}}$?

DEFINITION. Let $J$ and $J^0$ be Jacobi matrices and let $J^+$ be the Jacobi matrix whose off diagonal elements $a^+(i)$ are

$$a^+(i) = \begin{cases} a(i), & a(i) > a^0(i), \\ a^0(i), & a(i) \leqslant a^0(i), \end{cases} \quad i = 1, 2, \ldots \tag{III.2}$$

and whose diagonal elements $b^+(i)$ satisfy

$$b^+(i) = \begin{cases} b(i), & b(i) > b^0(i), \\ b^0(i), & b(i) \leqslant b^0(i), \end{cases} \quad i = 0, 1, 2, \ldots. \tag{III.3}$$

LEMMA (III.1). *Let* $\tau(J) \leqslant \tau(J^+) < \infty$ *and* $\lambda_0 \geqslant \tau(J^+)$. *Let* $N^+_{J^+}(\lambda_0) (N^+_J(\lambda_0))$ *be the number of eigenvalues of* $J^+ (J)$ *greater than* $\lambda_0$, *then* $N^+_J(\lambda_0) \leqslant N^+_{J^+}(\lambda_0)$.

*Proof.* Since $N^+_J(\lambda_0)$ is equal to the number of changes in sign $p(\lambda_0, n)$, $n = 0, 1, 2, \ldots$, and since $b^+(i) \geqslant b(i)$ and $a^+(i) \geqslant a(i)$, the result is a consequence of Sturm's comparison theorem (Fort [12, p. 152, Theorem 1]).

Let $\{\hat{p}^+(\lambda, n)\}$ be the scaled polynomials associated with $J^+$ (satisfying (II.14) and (II.15) but rescaled according to (II.16)). We now prove

LEMMA (III.2). *Suppose* $a^0(0) = 1$,

(i) $G_1(\lambda_0, i, k) > 0,\ k > i \geqslant -1,$

(ii) $G_1(\lambda_0, i, k) \leqslant G_1(\lambda_0, l, k) \leqslant G_1(\lambda_0, -1, k),\ l \leqslant i,$

(iii) sign $\hat{p}^+(\lambda_0, j) = \text{constant}$, $m < j < n < \infty$,

(iv) $\hat{p}^+(\lambda_0, n) = 0$ *or* sign $\hat{p}^+(\lambda_0, j) = \text{sign } \hat{p}^+(\lambda_0, n) = -\text{sign } \hat{p}^+(\lambda_0, n+1)$, $m < j < n$, *and*

(v) $\hat{p}^+(\lambda_0, m) = 0$ *or* $-\text{sign } \hat{p}^+(\lambda_0, m-1) = \text{sign } \hat{p}^+(\lambda_0, m) = \text{sign } \hat{p}^+(\lambda_0, j)$, $m < j < n$. *Then*

$$1 \leqslant \sum_{i=m}^{n-1} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| \right\} a^0(i+1)\, G_1(\lambda_0, -1, i). \tag{III.4}$$

*Proof.* The proof breaks up into two cases: Case 1, $\hat{p}^+(\lambda_0, m) = 0$, and Case 2, sign $\hat{p}^+(\lambda_0, m) = -\text{sign } \hat{p}^+(\lambda_0, m-1)$.

*Case* 1. Using (III.2) and (III.3) in (II.18) yields

$$\begin{aligned} \frac{\hat{p}^+(\lambda_0, k)}{\hat{p}^+(\lambda_0, m+1)} = {} & a^0(m+1)\, G_1(\lambda_0, m, k) \\ & - \sum_{i=m+1}^{k-1} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| G_1(\lambda_0, i, k) \right. \\ & \left. + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| G_1(\lambda_0, i+1, k) \right\} \\ & \times a^0(i+1) \frac{\hat{p}^+(\lambda_0, i)}{\hat{p}^+(\lambda_0, m+1)}. \end{aligned} \tag{III.5}$$

Since $\hat{p}^+(\lambda_0, k)/\hat{p}^+(\lambda_0, m+1) \geqslant 0$, $m+1 \leqslant k < n$, (III.5) implies that $\hat{p}^+(\lambda_0, k)/\hat{p}^+(\lambda_0, m+1) \leqslant a^0(m+1)\, G_1(\lambda_0, m, k)$, $m+1 \leqslant k < n$. Now substituting these results into (III,5) then using the fact that $\hat{p}^+(\lambda_0, n)/\hat{p}^+(\lambda_0, m+1) \leqslant 0$ ((iii) and (iv)) yields

$$\begin{aligned} G_1(\lambda_0, m, n) \leqslant & \sum_{i=m+1}^{n-1} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| G_1(\lambda_0, i, n) \right. \\ & \left. + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| G_1(\lambda_0, i+1, n) \right\} a^0(i+1)\, G_1(\lambda_0, m, i). \end{aligned} \tag{III.6}$$

It follows from (ii) above that we can replace $G_1(\lambda_0, i, n)$ and $G_1(\lambda_0, i+1, n)$ by $G_1(\lambda_0, m, n)$, which can then be eliminated from (III.6). Now replacing $G_1(\lambda_0, m, i)$ by $G_1(\lambda_0, -1, i)$ using (ii) gives the result.

*Case* 2. In this case we begin with

$$\frac{\hat{p}^+(\lambda_0, k)}{\hat{p}^+(\lambda_0, m)} = a^0(m)\, G_1(\lambda_0, m-1, k) - \frac{a^+(m)^2}{a^0(m)} \frac{\hat{p}^+(\lambda_0, m-1)}{\hat{p}^+(\lambda_0, m)} G_1(\lambda_0, m, k)$$
$$- \sum_{i=m}^{k-1} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| G_1(\lambda_0, i, k) \right.$$
$$\left. + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| G_1(\lambda_0, i+1, k) \right\} a^0(i+1) \frac{\hat{p}^+(\lambda_0, i)}{\hat{p}^+(\lambda_0, m)}. \tag{III.7}$$

Since $\hat{p}^+(\lambda_0, m-1)/\hat{p}^+(\lambda_0, m) < 0$ we have from above that

$$\frac{\hat{p}^+(\lambda_0, k)}{\hat{p}^+(\lambda_0, m} \leqslant \left( a^0(m) - \frac{a^+(m)^2}{a^0(m)} \frac{\hat{p}^+(\lambda_0, m-1)}{\hat{p}^+(\lambda_0, m)} \right) G_1(\lambda_0, m-1, k),$$

where (ii) has been used. Again using the fact that $\hat{p}^+(\lambda_0, n)/\hat{p}^+(\lambda_0, m) \leqslant 0$ and (ii) above in (III.7) yields

$$G_1(\lambda_0, m, n) \leqslant \sum_{i=m}^{n-1} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| G_1(\lambda_0, i, n) \right.$$
$$\left. + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| G_1(\lambda_0, i+1, n) \right\} a^0(i+1)\, G_1(\lambda_0, m-1, i). \tag{III.8}$$

The result now follows by using (ii) once again.

THEOREM (III.1). *Given $J$ choose $J^0$ such that $\tau(J) \leqslant \tau(J^0) = \tau(J^+)$, where $J^+$ is given by* (III.2) *and* (III.3). *Suppose that for $\lambda_0 \geqslant \tau(J^0)$, $0 < G_1(\lambda_0, n, m) \leqslant G_1(\lambda_0, n, m) \leqslant G_1(\lambda_0, -1, m)$, $-1 \leqslant k \leqslant n < m$, with $a^0(0) = 1$. Then*

$$N_J^+(\lambda_0) \leqslant N_{J^+}^+(\lambda_0)$$
$$\leqslant \sum_{i=0}^{\infty} \left\{ \left| \frac{b^+(i) - b^0(i)}{a^0(i+1)} \right| + \left| 1 - \frac{a^+(i+1)^2}{a^0(i+1)^2} \right| \right\} a^0(i+1)\, G_1(\lambda_0, -1, i).$$

*Proof.* The theorem follows from Lemma (III.1) and Lemma (III.2).

COROLLARY (III.1). *Given $J$ choose $J^0$ such that $\rho(J) \geqslant \rho(J^0) = \rho(J^-)$, where the coefficients of $J^-$ are chosen as $a^-(i) = a^+(i)$ and*

$$b^-(i) = \begin{cases} b^0(i), & b(i) \geqslant b^0(i) \\ b(i), & b(i) < b^0(i). \end{cases}$$

*Furthermore suppose that for* $\lambda_0 \leqslant \rho(J^0)$, $0<|G_1(\lambda_0, n, m)| \leqslant |G_1(\lambda_0, k, m)| \leqslant |G_1(\lambda_0, -1, m)|$, $-1 \leqslant k \leqslant n < m$, *with* $a^0(0)=1$. *Let* $N_J^-(\lambda_0)$ *denote the number of eigenvalues of J less than* $\lambda_0$, *then*

$$N_J^-(\lambda_0) \leqslant N_{J^-}^-(\lambda_0) \leqslant \sum_{i=0}^{\infty} \left\{ \left| \frac{b^-(i)-b^0(i)}{a^0(i+1)} \right| + \left| 1 - \frac{a^-(i+1)^2}{a^0(i+1)^2} \right| \right\} a^0(i+1)\, |G_1(\lambda_0, -1, i)|.$$

*Proof.* Setting $p_-(\lambda, n) = (-1)^n \ p(-\lambda, n)$ and $p^0(\lambda, n) = (-1)^n \ p^0(-\lambda, n)$ in (II.14) and (II.1), respectively, then using Theorem (III.1) gives the result.

EXAMPLE (III.1) (Tchebychev). Setting $a^0(n)=\frac{1}{2}$ and $b^0(n)=0$, one finds

$$G_1(\lambda, n, m) = \begin{cases} 0, & n \geqslant m \\ 2\dfrac{(z^{m-n} - z^{-(m-n)})}{z - 1/z}, & -1 \leqslant n < m \end{cases} \tag{III.9}$$

with $z=\lambda-\sqrt{\lambda^2-1}$. Equation (II.19) becomes with $j=0$

$$\begin{aligned} \psi(z, m) = {} & \frac{1-z^{2(m+1)}}{1-z^2} \\ & + \sum_{n=0}^{m-1} \left\{ (1-4a(n+1)^2) \left( \frac{1-z^{2(m-n-1)}}{1-z^2} \right) \right. \\ & \left. - 2b(n) \left( \frac{1-z^{2(m-n)}}{1-z^2} \right) \right\} \psi(z, n), \end{aligned} \tag{III.10}$$

where $\hat{\psi}(z, n) = z^n \hat{p}(\lambda, n)$. From Theorem (III.1) one finds

$$N_{J^+}^+(\lambda_0) \leqslant \sum_{n=0}^{\infty} \{ |1-4a^+(n+1)^2|\, z_0^2 + 2\,|b^+(n)|\, z_0 \} \frac{1-z_0^{2(n+1)}}{1-z_0^2} \tag{III.11}$$

for $\lambda_0 \geqslant 1$ $(z_0 \leqslant 1)$. Here $b^+(n) \geqslant 0$ and $a^+(n) \geqslant \frac{1}{2}$. Of course the sum will diverge unless $\limsup a(n) \leqslant \frac{1}{2}$ and $\limsup b(n) \leqslant 0$. Setting $\lambda_0 = z_0 = 1$ gives the result found in Geronimo [13].

EXAMPLE (III.2) (Shifted Tchebychev). Suppose $a^0(n)=\alpha>0$ and $b^0(n)=\beta$, then $G_1(\lambda_0, n, m)$ is the same as in (III.9) except that in this case

$$z = \frac{\lambda-\beta}{2\alpha} - \sqrt{\left(\frac{\lambda-\beta}{2\alpha}\right)^2 - 1}.$$

In this case $\sigma(J_0) = [\beta - 2\alpha, \beta + 2\alpha]$ and one finds

$$N_{J^+}^+(\lambda_0) \leqslant \sum_{i=0}^{\infty} \gamma_+(z_0, i)\left(\frac{1 - z_0^{2(i+1)}}{1 - z_0^2}\right), \tag{III.12}$$

where $\gamma_+(z_0, i) = |1 - a^+(i+1)^2/\alpha^2|\; z_0^2 + |(b^+(i) - \beta)/\alpha|\; z_0$, and $\lambda_0 \geqslant \beta + 2\alpha$ ($z_0 \leqslant 1$). Note that if lim sup $a(i) < \alpha$ and lim sup $b(i) < \beta$, there will only be a finite number of terms in (III.12). Furthermore the above formula applies even if $a(i) \to 0$.

EXAMPLE (III.3) (Unbounded case, Laguerre polynomials). If $a^0(n) = (n(n+\alpha))^{1/2}$ and $b^0(n) = -(2n + 1 + \alpha)$, the solutions to (II.1) and (II.2) are

$$p^\alpha(x, n) = \binom{n+\alpha}{n}^{1/2} L_n^\alpha(-x), \qquad \alpha > -1, \tag{III.13}$$

where the $\{p^\alpha(\lambda, n)\}$ are orthonormal with respect to the weight $((-x)^\alpha e^x/\Gamma(\alpha+1))\, dx$, $x \leqslant 0$, i.e.,

$$\int_{-\infty}^{0} p^\alpha(x, n)\, p(x, m)\, \frac{e^x(-x)^\alpha}{\Gamma(\alpha+1)}\, dx = \delta_{n,m}.$$

These polynomials have the following representation in terms of hypergeometric functions (Szegö [22, p. 103]):

$$p^\alpha(x, n) = \binom{n+\alpha}{n}^{1/2} {}_1F_1(-n, \alpha+1, -x). \tag{III.14}$$

The functions of the second kind $Q^\alpha(x, n)$ have the representation (Lebedev [16, p. 268])

$$\begin{aligned} Q^\alpha(x, n) &= \int_{-\infty}^{0} \frac{p^\alpha(t, n)}{x - t}\, \frac{e^t(-t)^\alpha}{\Gamma(\alpha+1)}\, dt, && n \geqslant 0, \\ &= x^\alpha\, \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \binom{n+\alpha}{n}^{1/2} \psi(n+\alpha+1, \alpha+1, x), && |\arg x| < \Pi, \end{aligned} \tag{III.15}$$

where $\psi(n+\alpha+1, \alpha+1; x)$ is the confluent hypergeometric function of the second kind. A representation for the Green's function now follows from (II.11) and in particular for $\alpha \neq 0$

$$G(0,n,m)=\begin{cases}0, & n>m\\ \dfrac{1}{\alpha}\sqrt{\dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\dfrac{\Gamma(m+\alpha+1)}{\Gamma(m+1)}} & \\ \times\left[\dfrac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}-\dfrac{\Gamma(m+1)}{\Gamma(m+\alpha+1)}\right], & 0\leqslant n<m.\end{cases} \qquad \text{(III.16)}$$

Furthermore from (II.13) we have, with $a^0(0)\equiv 1$,

$$G_1(0,-1,m)=\left(\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(m+1)}\right)^{1/2}, \qquad -1<m. \qquad \text{(III.17)}$$

Thus if $\alpha\geqslant 1$, (ii) of Lemma (III.3) is satisfied and in particular for $\alpha=1$,

$$N^+_{J^+}(0)\leqslant\sum_{i=0}^{\infty}\left\{\left|\frac{b^+(i)+2(i+1)}{i+1}\right|+\left|1-\frac{a^+(i+1)^2}{i+2(i+1)}\right|\right\}(i+2)^{3/2}.$$

## IV. The Birman–Schwinger Bound

As mentioned in the Introduction another bound on the number of eigenvalues of a Jacobi matrix may be obtained using the Birman–Schwinger argument. This bound has the advantage of being applicable even when the coefficients in the Jacobi matrix are themselves finite matrices (see below). The Birman–Schwinger argument uses the following max–min theorem (Reed and Simon [19, Theorem XIII.1]).

Theorem IV.1 (max–min principle). *Let $J$ be a self-adjoint operator that is bounded from above, i.e., $J\leqslant cI$ for some $c<\infty$. Set*

$$\mu_n=\inf_{\varphi_1,\varphi_2\cdots\varphi_{n-1}} U_J(\varphi_1,\varphi_2\cdots\varphi_{n-1}), \qquad \varphi_i\in D(J),$$

*where*

$$U_J(\varphi_1,\varphi_2\cdots\varphi_k)$$
$$=\sup\,\langle\psi,J\psi\rangle, \qquad \psi\in D(J),\ \|\psi\|=1,\ \langle\psi,\varphi_i\rangle=0,\ i=1,2\cdots k.$$

*Then, for each fixed n, either*

(a) *there are n eigenvalues (counting multiplicity) above the top of the essential spectrum and $\mu_n$ is the nth eigenvalue, or*

(b) *$\mu_n$ is the top of the essential spectrum and in that case $\mu_n = \mu_{n+1} = \mu_n + \cdots$ and there are at most $n-1$ eigenvalues (counting multiplicity) above $\mu_n$.*

This theorem has an important consequence that we will use later.

THEOREM (IV,2). *Let $J \leqslant 0$ and $J_p$ be self-adjoint operators. Let $J_p$ be compact and $0 \in \sigma_{ess}(J)$. Then $\mu_n(J+\beta J_p)$ is a continuous non-decreasing function of $\beta$ for $\beta \geqslant 0$ and strictly monotone once $\mu_n$ becomes positive.*

*Proof.* By the above hypothesis on $J_p$ the operator $J+\beta J_p$ is self-adjoint on $D(J)$ and $\sigma_{ess}(J+\beta J_p) = \sigma_{ess}(J)$ (Kato [15, p. 244]) for all $\beta$. Since $\mu_n(J+\beta J_p) \geqslant 0$ for all $n$, we have from the max–min principle that

$$\mu_n(J+\beta J_p) = \min_{\varphi_1, \varphi_2 \cdots \varphi_{n-1}} \max[g_\psi(\beta)],\ \psi \in D(J),\ \psi_i \in D(J),\ \|\psi\| = 1,$$
$$\langle \psi, \varphi_i \rangle = 0,\ i = 1, 2 \cdots n-1,$$

where $g_\psi(\beta) = \max[0, \langle \psi, J+\beta J_p \psi \rangle]$. Since $J \leqslant 0$, for fixed $\psi$, $g_\psi(\beta)$ is either zero or a strictly increasing function of $\beta$, furthermore $g_\psi(\beta)$ is a continuous function of $\beta$. Because $J_p$ is compact we find that for all $\psi$, $|\langle \psi, J_p \psi \rangle| \leqslant m^2 \langle \psi, \psi \rangle$, where $m$ is the norm of $J_p$. Thus if $|\beta_1 - \beta_2| \leqslant \delta/m^2$ then

$$|g_\psi(\beta_1) - g_\psi(\beta_2)| \leqslant |\beta_1 - \beta_2|\ |\langle \psi, J_p \psi \rangle| \leqslant |\beta_1 - \beta_2|\ m^2 < \delta,$$

showing that $g_\psi(\beta)$ is equicontinuous in $\psi$ yielding the result.

We now construct the resolvent operator $R^0(x)$ by solving the equation

$$(J^0 - \lambda I)\, R^0(\lambda) = I = R^0(\lambda)(J^0 - \lambda I), \tag{IV.1}$$

where $J^0$ is self-adjoint. By definition $R^0(\lambda)$ is well defined for $\lambda \notin \sigma_{dis}(J^0)$ and for $\lambda \notin \sigma(J^0)$, $R^0$ is a bounded operator. For Jacobi matrices we have the following representation for $R^0(\lambda)$ (Case [5], Case and Kac [6], Wall [23, p. 229]).

LEMMA (IV.1).

$$R^0(\lambda, n, m) = \begin{cases} -Q^0(\lambda, n)\, p^0(\lambda, m), & n > m \\ -Q^0(\lambda, m)\, p^0(\lambda, n), & 0 \leqslant n < m, \end{cases} \tag{IV.2}$$

*where $R^0(\lambda, n, m)$ is the $(n+1, m+1)$ matrix element of $R^0(\lambda)$, $\{Q^0(\lambda, n)\}$ are the functions of the second kind* (see (II.9)), *and $\{p^0(\lambda, n)\}$ are the orthonormal polynomials associated with $J^0$.*

*Proof.* Since the inverse is unique for $x \notin \sigma(J^0)$ we need only demonstrate that (IV.2) satisfies the necessary conditions. From the lefthand side of (IV.1) we find that $R^0(\lambda, n, m)$ satisfies (II.3) for $n \geqslant 0$ and $m \geqslant 0$, where we take $R^0(\lambda, -1, m) = 0 = R^0(\lambda, n, -1)$. Now (II.9) and the fact that $W[Q, P] = -1$ imply that the representation given by (IV.2) satisfies (II.3). That the right-hand side of (IV.1) is satisfied follows from the symmetry of $n$ and $m$ in (IV.2). Finally the fact that $\{Q^0(\lambda, n)\} \in l_2$, $\lambda \notin \sigma(J)$ implies that $R^0(\lambda)$ is a bounded operator for $\lambda \notin \sigma(J)$. Now we prove

THEOREM (IV.3). *Let $J: D(J) \to l_2$, $D(J) \subset l_2$, be a self-adjoint operator, and suppose that $J = J^0 + J_1$, where $J^0$ is self-adjoint and $J_1 = J - J^0$ is compact. Suppose furthermore that $\sigma(J^0) \supset (c, b]$, with $b \in \sigma_{\text{ess}}(J^0)$, and $b < \infty$, then for $\lambda_0 > b$,*

$$N_J^+(\lambda_0) \leqslant \operatorname{tr}[J_1 R^0(\lambda_0)]^2.$$

*Proof.* If $\operatorname{tr}[J_1 R^0(\lambda_0)]^2 = \infty$ there is nothing to prove, so suppose $\operatorname{tr}[J_1 R^0(\lambda_0)]^2 < \infty$. We wish to find an upper bound on the number of $l_2$ solutions of

$$(J - \lambda I)\,\psi = 0, \qquad \psi \in D(J) \tag{IV.3}$$

for $\lambda \geqslant \lambda_0$ and we begin by considering the operator $J^0 + \beta J_1$. Since $J_1$ is compact $D(J^0 + \beta J_1) = D(J^0)$ for all $\beta$ finite and we search for the $l_2$ solutions of

$$(J^0 + \beta J_1 - \lambda I)\,\psi = 0, \qquad \psi \in D(J^0) \tag{IV.4}$$

for $\lambda > \lambda_0$. For $\beta = 0$ there are no $l_2$ solutions to the above equation since $\lambda$ is above the spectrum of $J^0$, while for $\beta = 1$ the above operator is equal to $J$. Consequently $N_J^+(\lambda_0) =$ number of $\lambda_n(1) > \lambda_0$, where $\lambda_n(\beta)$ is an eigenvalue of (IV.4). From Lemma (IV.1), $\lambda_n(\beta)$ is a continuous monotone increasing function of $\beta$. Consequently $\lambda_n(1) > \lambda_0$ if and only if $\lambda_n(\beta) = \lambda_0$ for $0 < \beta < 1$. Labelling the particular value of $\beta$ for which $\lambda_n(\beta) = \lambda_0$, $\beta_n$, we see that there is only one $\beta_n$ for each $\lambda_n$. Thus $N_J^+(\lambda_0) \leqslant \sum_n 1/\beta_n^2$, $0 < \beta_n < 1$.

Since $R^0(\lambda_0)$ is negative definite there exists a self-adjoint operator $\hat{R} = (-R^0(\lambda_0))^{1/2}$ (Rudin [20, p. 349]). With $\hat{R}$ one can rewrite (IV.4) with $\lambda = \lambda_0$ as the discrete integral operator equation

$$K(\lambda_0) = (1/\beta)\,\varphi, \tag{IV.5}$$

where $K = \hat{R} J_1 \hat{R}$ and $\varphi = \hat{R}^{-1}\psi$. Since $\operatorname{tr} KK^* = \operatorname{tr}[J_1 \hat{R}^2]^2 = \operatorname{tr}[J_1 R^0(\lambda_0)]^2 < \infty$ by hypothesis, $K$ is a Hilbert–Schmidt operator. Consequently from the theory of integral equations (Widom [24])

$$\sum 1/\hat{\beta}_i^2 = \operatorname{tr}[J_1 R^0]^2,$$

where $\hat{\beta}_i$ is an eigenvalue of (IV.5). The result now follows by observing that the set $\{\beta_n\}$ is a subset of $\{\hat{\beta}_i\}$.

*Remark* (IV.1). If the point $b$ is not an eigenvalue of $J^0$, $R^0(b)$ is still a well-defined operator although now unbounded, and one can extend the above theorem to $\lambda_0 \geqslant b$.

In some cases Theorem (IV.5) gives a better bound than Theorem (III.1) as one approaches $\sigma_{\text{ess}}(J)$. This is especially true if the coefficients in the recurrence formula oscillate about their asymptotic values.

If $|R(\lambda_0, m, k)|$ decreases as we move away from the diagonal we have

$$N_J^+(\lambda_0) \leqslant \left\{ \sum_{n=0}^{\infty} |a(n+1) - a^0(n+1)| \, (|R^0(\lambda_0, n+1, n+1)| + |R^0(\lambda_0, n, n))| \right.$$
$$\left. + |b(n) - b^0(n)| \, |R^0(\lambda_0, n, n)| \right\}^2.$$

Example (IV.1) (matrix orthogonal polynomials). Let $l_2^p$ denote the Hilbert space of vectors $w = (w_i, ..., w_p)$, where $w_i \in l_2$. The scalar product on $l_2^p$ is the natural one $(f, g) = \sum_{i=1}^p (f_i, g_i)$, where $(f_i, g_i)$ is the scalar product in $l_2$. Let $e_i^p = (e_i, e_{i+1} \cdots e_{i+p-1})$, where $e_i$ is the usual unit vector in $l_2$. Suppose $J\colon D(J) \to l_2^p$, $D(J) \subset l_2^p$, is a self-adjoint operator with the representation

$$Je_{np}^p = A(n+1)\, e_{(n+1)p}^p + B(n)\, e_{np}^p + A(n)\, e_{(n-1)p}^p \tag{IV.6}$$

and

$$Je_0^p = A(1)\, e_p^p + B(0)\, e_0^p, \tag{IV.7}$$

where $A(n+1)$ and $B(n)$ are assumed to be $p \times p$ real symmetric matrices and $A(n+1) > 0$. We assume that $J = J^0 + J_1$, where $J_1 = J - J^0$ is a compact operator and $J^0$ is a self-adjoint operator satisfying (IV.6) and (IV.7) with $A(n+1)$ and $B(n)$ replaced by $A^0(n+1)$ and $B^0(n)$, respectively. Constructing the matrix polynomial solutions satisfying the equations

$$A^0(n+1)\, p^0(\lambda, n+1) + B^0(n)\, p^0(\lambda, n) + A^0(n)\, p^0(\lambda, n-1)$$
$$= \lambda p^0(\lambda, n), \qquad n = 0, 1, 2, ...$$

with the initial conditions

$$p^0(\lambda, 0) = I, \qquad p^0(\lambda, -1) = 0,$$

one finds that

$$\int p^0(\lambda, n)\, du^0\, p^0(\lambda, m)^+ = I\, \delta_{n,m},$$

where $A^+$ is the hermitian conjugate of $A$, $u^0$ is the spectral measure associated with $J^0$, and $I$ is the $p \times p$ identity matrix. Writing $Q^0(\lambda, n)$, the matrix function of the second kind, as

$$Q(\lambda, n) = \int \frac{p^0(x, n)}{\lambda - x} \, du^0, \qquad n \geqslant 0,$$

one has that the matrix analog to (IV.2) is

$$R^0(\lambda, n, m) = \begin{cases} -Q^0(\lambda, n)\, p^0(\lambda, m)^+, & n \geqslant m \\ -p^0(\lambda, n)\, Q^0(\lambda, m)^+, & 0 \leqslant n < m, \end{cases}$$

Assuming $\sigma(J^0) \subset (c, a]$, $a < \infty$, with $a \in \sigma_{\text{ess}}(J^0)$, Theorem (IV.3) yields for $\lambda_0 > a$ that

$$N_J^+(\lambda_0) \leqslant \operatorname{tr}(J_1 R^0(\lambda_0))^2 \leqslant \sum_{n,m=0} a(n, m)\, a(m, n),$$

where

$$\begin{aligned} a(n, m) = {} & |A(n+1) - A^0(n+1)|\, |R^0(\lambda_0, n+1, m)| \\ & + |B(n) - B^0(n)|\, |R^0(\lambda_0, n, m)| \\ & + |A(n) - A^0(n)|\, |R^0(\lambda_0, n-1, m)|. \end{aligned}$$

Here $|A| = \{\sum_{i,j} a_{i,j}^2\}^{1/2}$. In the special case where $A^0(n) = I/2$ and $B^0(n) = 0$ one finds

$$R^0(\lambda_0, n, m) = \begin{cases} -2z^{n+1} \left\{ \dfrac{z^{m+1} - z^{-m-1}}{z - 1/z} \right\} I, & n \geqslant m \\[2ex] -2z^{m+1} \left\{ \dfrac{z^{n+1} - z^{-n-1}}{z_0 - 1/z} \right\} I, & n < m \end{cases}$$

with $z = \lambda_0 - \sqrt{\lambda_0^2 - 1}$. Using the fact that for $\lambda_0 \geqslant 1$ $R(\lambda_0, n, m)$ decreases as we move away from the diagonal yields

$$N_J^+(\lambda_0) \leqslant 4p^2 \left\{ \sum_{i=0}^{\infty} |A(i+1) - \tfrac{1}{2}I| + |B(i)| \, \frac{1 - z^{2(i+1)}}{1 - z^2} \right\}^2.$$

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